Basic Elasticity - An Introductory Fragment

→ Elasticity is concerned with continuum dynamics of not-so-rigid bodies, i.e. bodies which give.

→ Slightly technical → little sense to grapple with details in 1 lecture

so

→ will skip many steps and focus on ideas and "big picture".

→ References for self-study, if interested

- Landau, Lifshitz: excellent, though terse (as usual)

- FW: 1 chapter (13) - a more detailed intro survey

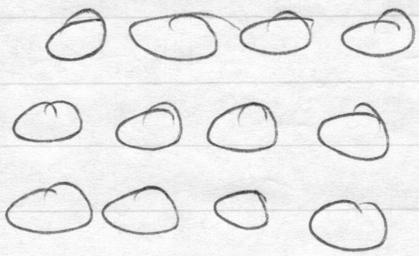
- Feynman Lectures: Vol II, online 2 (surprisingly) deep lectures

- Timoshenko: Formulary

→ Stress and Strain

Elasticity – "stuff"

elements



– deformation:

$$\underline{u}_i = \underline{x}'_i - \underline{x}_i \equiv \underline{u}(\underline{x})$$

after before displacement field

→ displacement field labels excursion as fctn. of position

→ subtle assumption of memory → each element knows where it started from.

– distance change upon deformation

initially $\xrightarrow{d\underline{x}_i}$ ⇒ distance for 2 pts. $d\underline{x}_i$

after deformation ⇒ $d\underline{x}_i + d\underline{u}_i$

$\xrightarrow[d\underline{x}_i + d\underline{u}_i]{}$

so length of separation changes.

- before $dl^2 = dx_i^2$ (sum patt.)

after $dl'^2 = (dx_i + du_i)^2$

- taking $du_i = \frac{\partial u_i}{\partial x_k} dx_k$

$\downarrow$

tensor.

then

$$dl'^2 = dl^2 + 2\frac{du_i}{dx_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l$$

and symmetrizing.

$$dl'^2 = dl^2 + 2u_{ik} dx_i dx_k$$

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}\right)$$

$\Downarrow$

Linear strain

NL

strain tensor

Key point: strain tensor $\Longleftrightarrow$ "gradient" of displacement field

tensor $\rightarrow$ off-diagonal components $\rightarrow$ shearing

$\rightarrow$ diagonal $\rightarrow$ compression, expansion

e.g. usually, concerned with linear elasticity, so

$$\boxed{u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)}$$

$\Rightarrow$ basic measure of perturbation

N.B.: $dV' = dV(1 + u_{ii})$
$= dV(1 + \mathrm{tr}\,\underline{\underline{u}})$

$\rightarrow$ Dynamics

$\rightarrow$ EOM

$\rightarrow$ Stress-Strain relation

→ EOM

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial}{\partial x_j} T_{ij} + \rho f_i$$

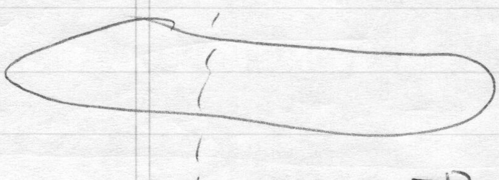
u_i → displacement

$\frac{\partial^2 u_i}{\partial t^2}$ → acceleration

T_{ij} → stress tensor (internal forces)

ρf_i → body force external (i.e. gravity)

stress tensor → net force/pressure transmitted thru (bounding) surface by medium



→ akin to Maxwell stress tensor in EM

→ $T \sim P$

→ T_{ij}? → Hooke's Law (yet again)

$T_{ij} \sim -E u_{ij}$ ⇒ $F = -kx$

E → restoring

u_{ij} → strain

both lowest order (linear) expansions. Spring constants ⇔ Youngs / Bulk Moduli

characterize medium. (ezbrm is [illegible])

Some tensor-ology:

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\underbrace{\delta_{ij}\,\mathrm{Tr}\,\underline{\underline{u}} = \delta_{ij}}_{\text{compression tied to } \mathrm{tr}\,\underline{\underline{u}}}\ \underline{\nabla}\cdot\underline{u}$$

Hooke's Law:

$$T_{ij} = -\overbrace{\lambda\,\delta_{ij}\,\mathrm{tr}\,\underline{\underline{u}}}^{\text{compression}} - \overbrace{2\mu\, u_{ij}}^{\text{includes shearing}}$$

$\rightsquigarrow$ $-$ $\Rightarrow$ restoring force

$\rightsquigarrow$ λ, μ $\Rightarrow$ Lamé coefficients.

(elastic moduli |

$\Rightarrow$ dimensionally F/a or E/V.

$$\underbrace{E/V}_{\text{"energy density"}}$$

$\Rightarrow$ medium spring constants

$\downarrow$

anisotropic

→ can re-write Hooke's Law as:

①

$$T_{ij} = -K\delta_{ij}\, \mathrm{tr}\,\underline{\underline{u}} - 2\mu\left(u_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{tr}\,\underline{\underline{u}}\right)$$

or

②

$$T_{ij} = -K\delta_{ij}\,\underline{\nabla}\cdot\underline{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\,\underline{\nabla}\cdot\underline{u}\right)$$

$$K = \lambda + 2/3\ \mu$$

and obviously: ①

$$\mathrm{tr}\,\underline{\underline{T}} = -3K\,\mathrm{tr}\,\underline{\underline{u}}$$

↓

bulk modulus

$$\boxed{K = -V\frac{\partial p}{\partial V}}$$

$$\frac{dV}{V} = \mathrm{tr}\,\underline{\underline{u}} = -\frac{dp}{K}$$

for uniform pressure applied to medium at rest

→ depends on eqn. of state

↕

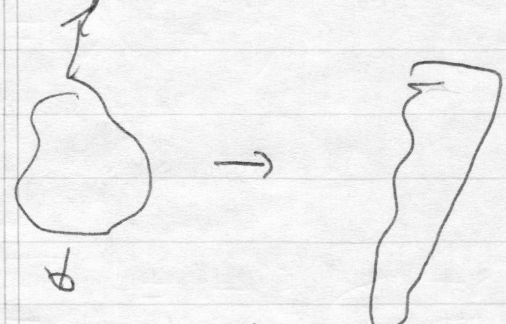
microscopics, etc.

and more crank:

$$u_{ij} = -\frac{1}{9K}\,\delta_{ij}\,\mathrm{tr}\,\underline{\underline{T}} - \frac{1}{2\mu}\left(\underline{\underline{T}}_{ij} - \frac{1}{3}\,\delta_{ij}\,\mathrm{tr}\,\underline{\underline{T}}\right)$$

strain → stress $\qquad \left(x = -\frac{f}{k}\right)$

→ Important example



medium with
uniform axial stress applied

i.e. $T_{zz} = p$, const

other $T_{ij} = 0 \qquad \begin{pmatrix} & 0 & 0 \\ 0 & 0 & \\ & & p \end{pmatrix}$

u_{ij} vs T_{ij} relation ⇝

tells how body will respond to stress.

Now,

$$u_{ij} = -\frac{1}{9K}\delta_{ij}\,\mathrm{tr}\,\underline{\underline{T}} - \frac{1}{2\mu}\left(\underline{\underline{T}}_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{tr}\,\underline{\underline{T}}\right)$$

$\Rightarrow$

$$u_{zz} = -\left(1/9K + \frac{1}{3\mu}\right)P$$

$$= -\left(1/9K + \frac{1}{3\mu}\right)T_{zz}$$

$u_{zz} < 0$ for $T_{zz} > 0$

usually put as

$$T_{zz} = -\underset{\substack{\uparrow \\ \text{young's modulus}}}{E}\, u_{zz}$$

$$E = \left(1/9K + 1/3\mu\right)^{-1}$$

$$= 9K\mu / 3K + \mu$$

but axial expansion $\rightarrow$ longitudinal compression

(i.e. this is response)

$$u_{xx} = u_{yy} = \left(\frac{1}{6\mu} - \frac{1}{9K}\right) P$$

$$= \left(\frac{1}{6\mu} - \frac{1}{9K}\right) T_{zz}$$

$u_{xx}, u_{yy} > 0$ for $T_{zz} > 0$

Note: $\begin{matrix} u_{xx} \\ u_{yy} \end{matrix} = -\sigma u_{zz}$

Poisson's ratio

$$\sigma = \frac{1}{2}\left(\frac{3K - 2\mu}{3K + \mu}\right)$$

$0 \rightarrow 1$

($1/2$ for $\underline{\nabla} \cdot \underline{u} = 0$)